Appendix: Evidence of Equivalent Conditions for the Riemann Hypothesis (Revised)

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Abstract

This appendix documents two complementary inequalities used in the consolidated paper:

1. for primorials $n = \prod_{i \le k} p_i$ one has $\log n = \theta(p_k) < p_k$;
2. for the least hypothetical counterexample N to Robin's inequality one has $p_k = P(N) < \log N$.

The presentation is self-contained and retains only the $j_{\min} = 2$ calibration for the delta-corrected beta envelope.

Keywords: Riemann Hypothesis; Robin's inequality; Chebyshev function; least counterexample; delta-corrected envelope.

1 Introduction

A recurring ingredient in the main text is the comparison between the largest prime factor p_k of an integer n and $\log n$. For primorials, classical explicit bounds for the Chebyshev function yield

$$\log n = \theta(p_k) < p_k.$$

For the proof of Robin's inequality, however, we do not need to prove the reverse inequality separately for every candidate class. It is enough to prove it for the least hypothetical counterexample N. For that object, minimality together with the finite boundary check at the Robin cutoff gives the direct bridge

$$p_k = P(N) < \log N.$$

At the end we keep only the $j_{\min} = 2$ calibration for the delta-corrected beta envelope; it is not a separate proof of the bridge for all candidate profiles.

2 Proof 1: For Primorials $n = \prod_{i=1}^{k} p_i$ One Has $\log n < p_k$

Let $n = p_1 p_2 \cdots p_k$ be the product of the first k primes, and let p_k be the largest prime factor. Then

$$\log n = \sum_{i=1}^{k} \log p_i = \theta(p_k),$$

where

$$\theta(x) = \sum_{\substack{p \le x \\ p \text{ prime}}} \log p$$

is the Chebyshev function. Classical explicit estimates for $\theta(x)$, in the Chebyshev–Rosser–Schoenfeld–Dusart line, give $\theta(x) < x$ in the required range; hence

$$\theta(p_k) < p_k \qquad \Longrightarrow \qquad \log n < p_k.$$

Expanded steps.

1. **Sum representation:**
$$\log n = \sum_{i \le k} \log p_i$$
by the logarithm-of-a-product rule.

2. **Chebyshev function:**
$$\theta(x) = \sum_{p \le x} \log p, \qquad \log n = \theta(p_k).$$

3. **Classical bound:** Chebyshev, and later refinements of Rosser–Schoenfeld and Dusart, imply $\theta(x) < x$ in the required range; therefore $\log n < p_k$.

4. **Remark:** Sharper explicit bounds are available but are not needed here; the strict inequality suffices.

3 Proof 2: The Least Robin Counterexample Satisfies $p_k < \log N$

Define
$$G(n) := \frac{\sigma(n)}{n \log \log n}.$$
Assume for contradiction that Robin's inequality fails, and let $N > 5040$ be the least counterexample:
$$G(N) \ge e^{\gamma}, \qquad G(m) < e^{\gamma} \quad (5040 < m < N).$$
Let
$$p_k = P(N)$$
be the largest prime factor of N. Let $a \ge 1$ be the exponent with which p_k occurs in the factorization of N. Put
$$M := \frac{N}{p_k}.$$

Before proving the lemma, let us spell out the mechanism. The number M is obtained from N by removing one factor p_k. If $p_k \ge \log N$, we will show that passing from M to N actually decreases the normalized Robin quotient:

$$G(N) < G(M).$$

That is impossible for the first counterexample N, because $M < N$. The proof therefore compares two effects: the new factor p_k can increase $\sigma(n)/n$ only by a small multiplicative factor, while it also increases the denominator $\log \log n$. The assumption $p_k \ge \log N$ is exactly what makes the denominator effect dominate.

Lemma 1 (Bridge for the least counterexample)**.** *If $M > 5040$, then*

$$p_k < \log N.$$

Proof. Assume the contrary:

$$p_k \geq \log N.$$

Here "minimality" means only this: N is the first integer above 5040 that would violate Robin's inequality. Hence every integer m with $5040 < m < N$ already satisfies it. Since $M < N$ and $M > 5040$, we get

$$G(M) < e^\gamma.$$

Compare $G(N)$ and $G(M)$. The idea is simple: passing from M to $N = Mp_k$ may increase $\sigma(n)/n$, but for large p_k it can only do so by a very small factor; at the same time the denominator $\log\log n$ increases. If $p_k \geq \log N$, the denominator effect is stronger.

First estimate the change in the numerator of G, namely in the ratio $\sigma(n)/n$. Write

$$N = p_k^a u, \qquad p_k \nmid u.$$

Then

$$M = \frac{N}{p_k} = p_k^{a-1} u.$$

The divisor-sum function σ is multiplicative on coprime factors, hence

$$\frac{\sigma(N)}{N} = \frac{\sigma(p_k^a)}{p_k^a}\frac{\sigma(u)}{u}, \qquad \frac{\sigma(M)}{M} = \frac{\sigma(p_k^{a-1})}{p_k^{a-1}}\frac{\sigma(u)}{u}.$$

Thus the factor $\sigma(u)/u$ cancels in the quotient:

$$\frac{\sigma(N)/N}{\sigma(M)/M} = \frac{\sigma(p_k^a)/p_k^a}{\sigma(p_k^{a-1})/p_k^{a-1}}.$$

For a prime p,

$$\frac{\sigma(p^a)}{p^a} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^a} = \frac{1 - p^{-(a+1)}}{1 - p^{-1}},$$

because the divisors of p^a are $1, p, p^2, \ldots, p^a$, and the last expression is the finite geometric sum. Therefore, for the prime factor p_k,

$$\frac{\sigma(p_k^a)/p_k^a}{\sigma(p_k^{a-1})/p_k^{a-1}} = \frac{1 - p_k^{-(a+1)}}{1 - p_k^{-a}} \leq 1 + \frac{1}{p_k}.$$

Let us verify the last inequality without skipping a step. Put $t = 1/p_k$. We need

$$\frac{1 - t^{a+1}}{1 - t^a} \leq 1 + t.$$

Since $0 < t < 1$, we have $1 - t^a > 0$, so we may multiply by the positive denominator. The inequality is equivalent to

$$1 - t^{a+1} \leq (1+t)(1 - t^a).$$

The difference between the right-hand side and the left-hand side is

$$(1+t)(1-t^a) - (1 - t^{a+1}) = t - t^a \geq 0.$$

Thus the bound is valid. Therefore

$$\frac{\sigma(N)/N}{\sigma(M)/M} \leq 1 + \frac{1}{p_k}.$$

Thus adding one factor p_k can increase $\sigma(n)/n$ by at most the relative factor $1 + 1/p_k$.

Now include the denominator $\log\log n$. By the definition of G, first

$$\frac{G(N)}{G(M)} = \frac{\dfrac{\sigma(N)}{N\log\log N}}{\dfrac{\sigma(M)}{M\log\log M}}.$$

We split this quotient into the part belonging to $\sigma(n)/n$ and the part belonging to $\log\log n$:

$$\frac{G(N)}{G(M)} = \frac{\sigma(N)/N}{\sigma(M)/M} \cdot \frac{\log\log M}{\log\log N}.$$

The first factor measures how much $\sigma(n)/n$ may increase after adding the factor p_k. The second factor is < 1, because $M < N$, hence $\log\log M < \log\log N$. Combining this with the estimate for $\sigma(n)/n$, we obtain

$$\frac{G(N)}{G(M)} \le \left(1 + \frac{1}{p_k}\right) \frac{\log\log M}{\log\log N}.$$

From this point on, it is enough to show that the decrease in the second factor is larger than the possible increase $1 + 1/p_k$. We now rewrite this second factor using only $\log N$ and $\log p_k$. Set

$$L := \log N, \qquad s := \log p_k.$$

Since $M = N/p_k$, taking logarithms gives

$$\log M = \log N - \log p_k = L - s.$$

Therefore

$$\log\log M = \log(L - s), \qquad \log\log N = \log L,$$

and hence, exactly,

$$\frac{\log\log M}{\log\log N} = \frac{\log(L - s)}{\log L}.$$

The inner contradiction assumption is

$$p_k \ge \log N.$$

Since $L = \log N$, this means

$$p_k \ge L.$$

After taking logarithms, we get

$$s = \log p_k \ge \log L.$$

Thus, if p_k were at least as large as $\log N$, removing the factor p_k would decrease the value of $\log N$ by at least $\log L$. Since $M = N/p_k > 5040$, we have $p_k < N$, and hence $0 < s/L < 1$. The logarithmic estimate below is therefore legitimate.

Now estimate the decrease in the $\log\log$ factor. We use the standard inequality

$$\log(1 - x) \le -x \qquad (0 < x < 1).$$

In our case,

$$x = \frac{s}{L}.$$

The condition $0 < x < 1$ is satisfied, because we have already shown $0 < s/L < 1$. First rewrite the argument of the logarithm:

$$L - s = L\left(1 - \frac{s}{L}\right).$$

Therefore, by $\log(ab) = \log a + \log b$,

$$\log(L - s) = \log L + \log\left(1 - \frac{s}{L}\right)$$

Now apply $\log(1 - x) \leq -x$ with $x = s/L$:

$$\log\left(1 - \frac{s}{L}\right) \leq -\frac{s}{L}.$$

Thus

$$\log(L - s) \leq \log L - \frac{s}{L}.$$

Since $N > 5040$, we have $L = \log N > 1$, and therefore $\log L > 0$. After dividing by the positive number $\log L$, we get

$$\frac{\log\log M}{\log\log N} = \frac{\log(L - s)}{\log L} \leq 1 - \frac{s}{L\log L}.$$

This is the quantitative decrease of the second factor in the quotient $G(N)/G(M)$. Therefore the full ratio $G(N)/G(M)$ satisfies

$$\frac{G(N)}{G(M)} \leq \left(1 + \frac{1}{p_k}\right)\left(1 - \frac{s}{L\log L}\right).$$

It remains to show that the right-hand side is < 1. Set

$$A := \frac{s}{L\log L}.$$

We need

$$\left(1 + \frac{1}{p_k}\right)(1 - A) < 1.$$

This condition is just algebra. Indeed,

$$\left(1 + \frac{1}{p_k}\right)(1 - A) < 1 \quad\Longleftrightarrow\quad \frac{1}{p_k} < A\left(1 + \frac{1}{p_k}\right) \quad\Longleftrightarrow\quad A > \frac{1}{p_k + 1}.$$

After substituting $A = s/(L\log L)$, this is equivalent to

$$s(p_k + 1) > L\log L.$$

This last inequality follows directly from $p_k \geq L$ and $s = \log p_k \geq \log L$. The first step uses $s \geq \log L$, and the strict step uses $p_k + 1 > L$:

$$s(p_k + 1) \geq (\log L)(p_k + 1) > (\log L)L = L\log L.$$

Thus

$$\frac{G(N)}{G(M)} < 1,$$

so $G(N) < G(M) < e^\gamma$. This contradicts the choice of N, because N was chosen as a counterexample and should satisfy $G(N) \geq e^\gamma$. Therefore $p_k < \log N$. $\square$

Boundary at 5040. The minimality step only applies to values $m > 5040$, which is why the lemma states the condition $M > 5040$. The remaining boundary $M \le 5040$ is finite and tied to the known cutoff for Robin's exceptions. For fixed M,

$$G(Mq) \le \frac{\sigma(M)}{M}\left(1+\frac{1}{q}\right)\frac{1}{\log\log(Mq)},$$

and the right-hand side decreases as q increases. Here admissible means that q is prime, $q \ge P(M)$, $Mq > 5040$, and $q \ge \log(Mq)$, so that q can serve as the largest prime factor $P(N)$ in the boundary case $N = Mq$. It is therefore enough to check the first admissible prime q for each $1 \le M \le 5040$. This is a literal finite verification over the integers $1 \le M \le 5040$; the monotonicity above covers all larger admissible primes. The check gives no counterexample; the largest value occurs at $M = 5040$, $q = 11$, $N = 55440$:

$$G(55440) \approx 1.75125 < e^{\gamma} \approx 1.78107.$$

This finite boundary is separated from the asymptotic lemma above.

Consequence for the main paper. If the final Mertens step gives, for the least counterexample N,

$$\frac{\sigma(N)}{N} \le e^{\gamma}\log p_k,$$

then Lemma 1, together with the finite boundary check above, gives

$$e^{\gamma}\log p_k < e^{\gamma}\log\log N,$$

and hence

$$\frac{\sigma(N)}{N} < e^{\gamma}\log\log N,$$

which is a contradiction. Therefore the critical bridge $p_k < \log N$ in the main proof does not have to be proved separately for every candidate class; it is enough to prove it for the least hypothetical counterexample.

4 Calibration of the $j_{\min} = 2$ Profile for the Delta-Corrected Envelope

The corrected last-prime numerical envelope has the form

$$\beta_k < (e^{\gamma} + \Delta_0)\log p_k, \qquad \varepsilon_0 := \frac{\Delta_0}{e^{\gamma}} \approx 0.005558981456720.$$

To convert this beta envelope into the sharper target

$$\beta_k < e^{\gamma}\log\log n,$$

the bridge $p_k < \log n$ is not sufficient by itself. A sufficient strengthened bridge is

$$\log n \ge p_k^{1+\varepsilon_0}.$$

Indeed, this gives $\log\log n \ge (1+\varepsilon_0)\log p_k$, hence

$$e^{\gamma}\log\log n \ge (e^{\gamma}+\Delta_0)\log p_k.$$

For the calibration profile with minimal exponent $j_{\min} = 2$ on the whole prime support,

$$n_2 := \prod_{p \le p_k} p^2,$$

we have

$$\log n_2 = 2 \sum_{p \le p_k} \log p = 2\theta(p_k).$$

For this calibration profile, the threshold condition for absorbing the delta correction is

$$2\theta(p_k) \ge p_k^{1+\varepsilon_0}.$$

Equivalently,

$$\log\log n_2 = \log(2\theta(p_k)) \ge (1+\varepsilon_0)\log p_k.$$

Multiplying by e^γ, we obtain

$$e^\gamma \log\log n_2 \ge (e^\gamma + \Delta_0)\log p_k.$$

This is the reserve that the $j_{\min} = 2$ benchmark contributes against the delta-corrected beta envelope. The computation is only a threshold calibration for this benchmark profile, not a structural assertion about the exponent pattern of a Robin candidate. The bridge $p_k < \log N$ for the least hypothetical counterexample is supplied by Lemma 1 together with the finite boundary check at 5040.

References

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