Moment–Centred Decomposition (MVDC): Central Bernoulli Numbers and High–Precision Euler–Product Corrections

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Abstract

The Mean–Value Decomposition by Centre (MVDC) separates a finite product into a chosen centre term and an exact residual. For smooth factor sequences, midpoint Taylor expansion of the logarithm is governed by centred powers of the index. These finite coefficients form a natural C–Bernoulli family and admit exact representations through central Bernoulli polynomials and a compact generating function. A second, distinct moment expansion follows from the Taylor series of $\log(1+x)$ when the factors, rather than their logarithms, are centred multiplicatively. Keeping these two constructions separate removes an ambiguity in an earlier draft while preserving the central Bernoulli structure of MVDC. The factorial application gives the usual exact Bernoulli coefficients around an analytic MVDC centre. For Euler products the finite prime block is evaluated exactly by its logarithmic mean, and only the remaining infinite tail must be modelled or bounded.

1 Exact decomposition and the role of the centre

Let

$$P_n = \prod_{i=1}^{n} a_i, \qquad a_i > 0, \qquad \ell_i := \log a_i.$$

For every real centre c, with $k = e^c$, define

$$H_c := k^n = e^{nc}, \qquad R_c(n) := \sum_{i=1}^{n} (\ell_i - c).$$

Then the fundamental MVDC identity is

$$\boxed{P_n = H_c\, e^{R_c(n)}, \qquad \log P_n = nc + R_c(n).} \tag{1}$$

No asymptotic assumption is used in (1). The centre may be selected analytically so that H_c absorbs known dominant terms, or numerically for stability.

There is one important limiting case. If

$$c = \mu_{\log} := \frac{1}{n} \sum_{i=1}^{n} \ell_i,$$

then $R_{\mu_{\log}}(n) = 0$ and $H_{\mu_{\log}} = P_n$ identically. This is a valid exact log-domain evaluation, but it leaves no nonzero residual to approximate. A nontrivial asymptotic application therefore uses a centre specified independently of the exact finite product, for example by an integral, an endpoint calculation, or known leading asymptotics.

2 The correct factor-moment expansion

Higher moments enter through the factors themselves. Write

$$a_i = k(1 + x_i), \qquad x_i = \frac{a_i}{k} - 1, \qquad X_r(k) := \sum_{i=1}^{n} x_i^r.$$

If $\rho := \max_i |x_i| < 1$, summing the Taylor series for $\log(1 + x_i)$ gives

$$\boxed{\log \frac{P_n}{k^n} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} X_r(k).} \tag{2}$$

After truncation at $r = M$, the remainder obeys the explicit bound

$$|\mathcal{R}_M| \le \sum_{i=1}^{n} \frac{|x_i|^{M+1}}{(M+1)(1 - |x_i|)}.$$

For the arithmetic-mean centre $k = \bar{a} = n^{-1} \sum_i a_i$, one has $X_1(k) = 0$, so the series starts at $r = 2$. This is the precise setting in which the cancellation of the first moment follows from $\log(1 + x)$. Logarithmic central moments $\sum(\ell_i - \mu_{\log})^r$ are useful descriptors of the data, but they are not interchangeable with the factor moments $X_r(k)$ in (2).

3 Central Bernoulli and Nörlund polynomials

Ordinary Bernoulli polynomials are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{r=0}^{\infty} B_r(x) \frac{t^r}{r!}.$$

The generalized Bernoulli polynomials of order α, often associated with Nörlund, are defined by

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{r=0}^{\infty} B_r^{(\alpha)}(x) \frac{t^r}{r!}. \tag{3}$$

One standard central Bernoulli family is

$$B_r^c(x) := 2^r B_r\left(\frac{x+1}{2}\right), \qquad \frac{t}{\sinh t} e^{xt} = \sum_{r=0}^{\infty} B_r^c(x) \frac{t^r}{r!}. \tag{4}$$

It satisfies $B_r^c(-x) = (-1)^r B_r^c(x)$. More generally,

$$\left(\frac{t}{\sinh t}\right)^{\alpha} e^{xt} = \sum_{r=0}^{\infty} B_{r,\alpha}^c(x) \frac{t^r}{r!}, \qquad B_{r,\alpha}^c(x) = 2^r B_r^{(\alpha)}\left(\frac{x+\alpha}{2}\right).$$

Thus central Bernoulli objects and their Nörlund connection are standard; the remaining question is how they enter a particular MVDC expansion.

4 Finite C–Bernoulli coefficients

Put $m_n = (n+1)/2$. The finite centred-index coefficients used in this note are

Definition 1 (Finite C–Bernoulli coefficients)**.**

$$\boxed{C_r(n) := \frac{1}{r!}\sum_{i=1}^{n}(i-m_n)^r, \qquad r \geq 0.} \tag{5}$$

This definition is purely algebraic and depends only on the geometry of the n equally spaced indices. Faulhaber's identity gives

$$C_r(n) = \frac{B_{r+1}\left(\frac{n+1}{2}\right) - B_{r+1}\left(\frac{1-n}{2}\right)}{(r+1)r!} = \frac{B^c_{r+1}(n) - B^c_{r+1}(-n)}{2^{r+1}(r+1)r!}. \tag{6}$$

The generating function is regular at the origin and has the compact form

$$\boxed{\sum_{r=0}^{\infty} C_r(n)t^r = \sum_{i=1}^{n} e^{(i-m_n)t} = \frac{\sinh(nt/2)}{\sinh(t/2)}.} \tag{7}$$

Symmetry now gives the correct parity:

$$C_{2j+1}(n) = 0.$$

The first nonzero coefficients are

$$C_0(n) = n, \qquad C_2(n) = \frac{n(n^2-1)}{24}, \qquad C_4(n) = \frac{n(n^2-1)(3n^2-7)}{5760}.$$

Theorem 1 (Midpoint expansion)**.** *Let $f \in C^{2p+2}[1,n]$. Then*

$$\sum_{i=1}^{n} f(i) = \sum_{j=0}^{p} f^{(2j)}(m_n)C_{2j}(n) + \mathcal{T}_{2p+2}, \tag{8}$$

where

$$|\mathcal{T}_{2p+2}| \leq \left\| f^{(2p+2)} \right\|_{\infty} C_{2p+2}(n).$$

Proof. Apply Taylor's theorem at m_n to every $f(i)$, sum the finite expansions, and use (5). All odd powers cancel by symmetry. Summing the Lagrange remainder bounds gives the stated estimate. □

For $f(i) = \log a_i$, Theorem 1 is the rigorous central Bernoulli bridge in MVDC. The $C_r(n)$ encode the centred indices, while the derivatives of the smooth interpolation $f(x)$ encode the particular factors. Consequently the same $C_r(n)$ may be reused for different smooth products without identifying them with the data-dependent moments of the values $f(i)$.

5 Euler–Maclaurin form and the factorial

For $g_c(x) = f(x) - c$, the Euler–Maclaurin formula on the indices $1, \ldots, n$ is

$$\begin{aligned}\sum_{i=1}^{n} g_c(i) = &\int_1^n g_c(x)\,dx + \frac{g_c(1)+g_c(n)}{2} \\ &+ \sum_{j=1}^{p} \frac{B_{2j}}{(2j)!}\left(g_c^{(2j-1)}(n) - g_c^{(2j-1)}(1)\right) + \mathcal{E}_p.\end{aligned} \tag{9}$$

The standard remainder $\mathcal{E}_p$ is an integral involving the periodic Bernoulli polynomial and $g_c^{(2p)}$. A centre can absorb some integral or boundary contributions in a specific application, but those cancellations must be checked for that application.

For the factorial, choose the analytic centre

$$c_n = \log n - 1 + \frac{\log(2\pi n)}{2n}, \qquad H_n = e^{nc_n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Euler–Maclaurin applied to $f(x) = \log x$, with the limiting constant fixed in the standard way, yields

$$\log \frac{n!}{H_n} \sim \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)n^{2j-1}}. \tag{10}$$

Thus

$$\log \frac{n!}{H_n} = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} + O(n^{-9}).$$

These are precisely the analytic coefficients implemented in `mvdc_factorial_analytic.py`. For this particular centre, the MVDC organization and the classical log–Stirling expansion are the same closed formula written as H_n times a small correction; this observation does not compare other MVDC centres or fitted cascades with Stirling expansions.

n	relative error after the n^{-7} term
10	8.23×10^{-13}
20	1.63×10^{-15}
50	4.31×10^{-19}
100	8.42×10^{-22}

Table 1: High-precision check of (10) using 80-digit arithmetic.

6 Application to truncated Euler products

Let

$$\zeta_N(s) = \prod_{p \le N} (1 - p^{-s})^{-1}, \qquad \sigma := \Re s > 1.$$

For $M > N$, define

$$f_s(p) := -\log(1 - p^{-s}), \qquad \mu_{N,M}(s) := \frac{1}{n} \sum_{N < p \le M} f_s(p), \qquad n = \pi(M) - \pi(N).$$

The finite block is exactly

$$B_{N,M}(s) := \sum_{N < p \le M} f_s(p) = n\mu_{N,M}(s).$$

Therefore the exact tail decomposition is

$$\log \frac{\zeta(s)}{\zeta_N(s)} = n\mu_{N,M}(s) + R_M(s), \qquad R_M(s) = \sum_{p > M} -\log(1 - p^{-s}). \tag{11}$$

The omitted tail has the elementary rigorous bound

$$|R_M(s)| \le \frac{M^{1-\sigma}}{(\sigma - 1)(1 - 2^{-\sigma})}. \tag{12}$$

Indeed, $-\log(1-p^{-\sigma}) \le p^{-\sigma}/(1-2^{-\sigma})$, and the prime sum is bounded by the corresponding integer integral.

For numerical acceleration one may model $R_M(2)$ by

$$I_M := \int_M^\infty \frac{x^{-2}}{\log x}\, dx.$$

This is a prime-density model, not the rigorous bound (12). With $M = 10N$, direct high-precision calculation gives:

N	n	exact tail Δ	$B_{N,M} + I_M$	error
1000	1061	$1.26989814756 \times 10^{-4}$	$1.27049267685 \times 10^{-4}$	5.95×10^{-8}
5000	4464	$2.10858622405 \times 10^{-5}$	$2.10884676662 \times 10^{-5}$	2.61×10^{-9}
10000	8363	$9.81570113247 \times 10^{-6}$	$9.81712884998 \times 10^{-6}$	1.43×10^{-9}

Table 2: Exact finite MVDC block plus the integral model for the remaining zeta tail.

The numerical accuracy in this table is genuine, but it comes from the exact finite block and the integral tail model. In the earlier experiment the additional logarithmic central-moment term at $N = 1000$ was -1.56×10^{-14}, far below the displayed 5.95×10^{-8} error. Such moments may still be studied as diagnostics, but they are not an additional term in (11).

6.1 Extension to Dirichlet L-functions

For a primitive Dirichlet character χ modulo q, define

$$f_{\chi,s}(p) := -\log(1-\chi(p)p^{-s}), \qquad p \nmid q.$$

Using a branch of the logarithm consistent with the absolutely convergent Euler product, the same exact decomposition holds:

$$\log \frac{L(s,\chi)}{L_N(s,\chi)} = n\mu_{N,M}(\chi,s) + R_{M,\chi}(s), \tag{13}$$

where n counts the primes $N < p \le M$ with $p \nmid q$. Since $|\chi(p)| \le 1$, the bound (12) also bounds $|R_{M,\chi}(s)|$.

For the nontrivial real character modulo 3,

$$\chi_3(n) = \begin{cases} 1, & n \equiv 1 \pmod 3, \\ -1, & n \equiv 2 \pmod 3, \\ 0, & 3 \mid n, \end{cases}$$

the reference value can be computed independently from

$$L(2,\chi_3) = \frac{1}{9}\left[\zeta\left(2,\frac{1}{3}\right) - \zeta\left(2,\frac{2}{3}\right)\right].$$

For $N = 10^3$ and $M = 10^6$, 60-digit arithmetic gives

$$\begin{aligned} &\text{exact missing tail} && -5.61728190475735 \times 10^{-8}, \\ &\text{exact finite block } (N,M] && -5.61013233973667 \times 10^{-8}, \\ &\text{omitted tail } p > M && -7.14956502068784 \times 10^{-11}. \end{aligned}$$

Thus the previously observed 10^{-10}-level agreement is retained, with its source identified correctly as the explicitly computed prime block rather than an added central-moment series.

7 Conclusion

The corrected analytic structure of MVDC has four distinct components:

- the exact centre decomposition (1), valid for every positive finite product;
- the convergent factor-moment expansion (2), with an explicit remainder;
- the finite C–Bernoulli coefficients (5), which encode centred index sums and enter smooth midpoint expansions through Theorem 1;
- application-specific analytic or numerical models for the residual, such as (10) and (11).

Central Bernoulli and Nörlund polynomials therefore remain a genuine part of the MVDC framework. What must not be conflated are centred index coefficients, moments of factor deviations, and central moments of logarithmic data. Once those objects are kept separate, the valid factorial and Euler-product computations remain intact and their error sources can be stated precisely.

References

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