

Candidate Cycles of the Collatz Conjecture: A Corrected Smooth Framework and the Exact Diophantine Bridge

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Abstract

We separate three objects that must not be conflated: arbitrary formal words in the affine maps $U(x) = 3x + 1$ and $H(x) = x/2$, positive p -vectors that prescribe candidate odd-to-odd blocks, and genuine Collatz-admissible blocks. We correct the extrema of the affine constant and replace the former correction term by a coherent deviation from the basic p -fractal. A cyclic-rotation identity then gives an exact Diophantine bridge: a positive p -vector closes to a positive Collatz cycle if and only if $D = 2^b - 3^a > 0$ divides its affine numerator $S(p)$.

The telescoping identity rigorously excludes all candidate words with no entry equal to 1, except for the trivial all-2 word. In the class $p_i \in \{1, 2\}$, we exclude the cases containing exactly one, two, or three entries equal to 1. We also prove that a proposed uniform cyclic size estimate cannot work: no fixed constant C can guarantee a rotation with $P < CD$. A largest-gap certificate and an exact identity between neighboring cuts survive as valid tools. The neighboring-quotient system is developed further through an exact product identity, a hierarchy of two-sided CRT windows, a deterministic partial state map for gaps in $\{2, 3\}$, and affine prime-power certificates, including a closed criterion modulo 5. We also prove effective finiteness of the cyclic two-run class $2^u 3^v$. Finally, we record the correct logarithmic identity for a hypothetical cycle and state the remaining arithmetic problems explicitly.

This paper does not prove the full Collatz conjecture, does not exclude all nontrivial cycles, and does not address unbounded trajectories. The previously established result for the single basic-fractal cyclic class is retained as an input and is not reproved here.

1 Scope and the three models

Let

$$U(x) = 3x + 1, \quad H(x) = \frac{x}{2}.$$

The accelerated odd Collatz map is

$$\mathcal{T}(n) = \frac{3n + 1}{2^{v_2(3n+1)}} \quad (n \in \mathbb{N}_{>0} \text{ odd}).$$

Definition 1 (Formal smooth word). *A formal smooth word is an arbitrary finite composition of U and H , acting on a real or rational variable. It need not respect parity. In particular, a word beginning with H , containing two consecutive occurrences of U , or applying U to an even intermediate value is not automatically a Collatz trajectory.*

Suppose that a formal word σ contains a occurrences of U and b occurrences of H . Its affine map has the form

$$F_\sigma(x) = \frac{3^a}{2^b}x + C(\sigma), \quad C(\sigma) = \sum_{k=1}^a 3^{a-k} 2^{-J_k(\sigma)}, \quad (1.1)$$

where $J_k(\sigma)$ is the number of occurrences of H following the k -th occurrence of U .

Proposition 2 (Correct extrema of the affine constant). *For fixed $a, b > 0$, over all formal words with these symbol counts,*

$$C_{\min}(a, b) = \frac{3^a - 1}{2^{b+1}} \quad \text{for } U^a H^b, \quad C_{\max}(a, b) = \frac{3^a - 1}{2} \quad \text{for } H^b U^a.$$

Proof. Every J_k lies between 0 and b , and 2^{-J_k} decreases as J_k increases. The simultaneous values $J_k = b$ occur for $U^a H^b$, while $J_k = 0$ for every k occurs for $H^b U^a$. Substitution into (1.1) gives the two geometric sums. $\square$

Corollary 3 (Formal fixed points form a finite set). *Put $D = 2^b - 3^a$. If $D > 0$, each formal word has the unique positive fixed point*

$$x_\sigma = \frac{2^b C(\sigma)}{D}.$$

For fixed a, b , these fixed points form a finite set contained in the closed interval

$$\left[\frac{3^a - 1}{2D}, \frac{2^{b-1}(3^a - 1)}{D} \right].$$

An integral member of this formal set is not, by that fact alone, a Collatz cycle.

Definition 4 (General p -vector). *A positive vector*

$$p = (p_1, \dots, p_a) \in \mathbb{N}_{>0}^a, \quad p_i \geq 1,$$

prescribes the candidate odd-to-odd word

$$UH^{p_1}UH^{p_2} \dots UH^{p_a}.$$

Write

$$b = \sum_{i=1}^a p_i, \quad s_0 = 0, \quad s_k = \sum_{i=1}^k p_i,$$

and define

$$S(p) = \sum_{k=0}^{a-1} 3^{a-1-k} 2^{s_k}. \quad (1.2)$$

The vector is only a candidate until its prescribed divisions and odd endpoints have been proved exact.

Definition 5 (Collatz-admissible and closed pair). *Let $n_0 \in \mathbb{N}_{>0}$ be odd. The pair (n_0, p) is Collatz-admissible if*

$$n_{i+1} = \frac{3n_i + 1}{2^{p_{i+1}}}, \quad i = 0, \dots, a-1,$$

produces a positive odd integer at every step. It is closed if $n_a = n_0$. For an admissible pair, $p_{i+1} = v_2(3n_i + 1)$ automatically.

Definition 6 (Basic p -fractal). *For $b \geq a$, the basic p -fractal is*

$$p_{\text{base}} = (\underbrace{1, \dots, 1}_{a-1}, b - a + 1).$$

Its affine numerator is

$$S(p_{\text{base}}) = \sum_{k=0}^{a-1} 3^{a-1-k} 2^k = 3^a - 2^a. \quad (1.3)$$

This odd-to-odd object is different from the formal word $U^a H^b$: the latter generally contains consecutive U 's and need not be Collatz-admissible.

Theorem 7 (Previously established basic-fractal input). *For integers $a \geq 1$ and $b \geq a$, let p_{base} be as in Definition 6. If (n_0, p_{base}) is Collatz-admissible and closed, then its positive orbit is the trivial 1-cycle. This result was established in the original version of Part I [1] and is retained here with that exact scope as an input; no claim about arbitrary p -vectors is included in it.*

Remark 8 (Status of the imported result). Theorem 7 is not reproved below. The new work begins when one passes from this one cyclic class to general positive p -vectors.

2 A coherent correction term

The two baselines $U^a H^b$ and p_{base} must not be mixed. For comparisons inside the p -vector model, define

$$\varepsilon(p) := S(p) - S(p_{\text{base}}) = \sum_{k=0}^{a-1} 3^{a-1-k} (2^{s_k} - 2^k). \quad (2.1)$$

Lemma 9 (Positivity and equality case). *For every positive p -vector with fixed a, b ,*

$$\varepsilon(p) \geq 0.$$

Equality holds exactly for $p = p_{\text{base}}$.

Proof. Since every $p_i \geq 1$, we have $s_k \geq k$, so every summand in (2.1) is nonnegative. Equality holds precisely when $s_k = k$ for $0 \leq k \leq a-1$, equivalently $p_1 = \cdots = p_{a-1} = 1$. The fixed total b then forces $p_a = b - a + 1$. $\square$

Proposition 10 (Contraction gives only a lower bound). *If a positive closed pair exists, then*

$$3^a < 2^b, \quad b \geq \lfloor a \log_2 3 \rfloor + 1.$$

Contraction alone does not imply equality in the second relation.

Proof. The affine numerator $S(p)$ is positive. The fixed-point equation derived below is $Dn_0 = S(p)$, so positivity requires $D = 2^b - 3^a > 0$. Taking logarithms gives the integer lower bound. No upper bound on b follows from this argument. $\square$

3 The exact odd-to-odd Diophantine bridge

For $q \geq 1$, let

$$A_q(x) = \frac{3x+1}{2^q}, \quad F_p = A_{p_a} \circ \cdots \circ A_{p_1}.$$

Proposition 11 (Affine composition formula). *For every positive p -vector,*

$$F_p(x) = \frac{3^a x + S(p)}{2^b}. \quad (3.1)$$

Proof. The formula is immediate for one step. If the numerator after r steps is

$$3^r x + \sum_{k=0}^{r-1} 3^{r-1-k} 2^{s_k},$$

then applying $A_{p_{r+1}}$ multiplies that numerator by 3, adds 2^{s_r} , and changes the denominator to 2^{s_r+1} . Induction proves (3.1). $\square$

Proposition 12 (Formal positive fixed point). *Let*

$$D := 2^b - 3^a.$$

The affine map F_p has a positive real fixed point if and only if $D > 0$. It is unique and equals

$$x_p = \frac{S(p)}{D}.$$

It is a positive integer if and only if $D > 0$ and $D \mid S(p)$.

Definition 13 (Cyclic rotation). *For $p = (p_1, p_2, \dots, p_a)$, set*

$$\text{rot } p = (p_2, \dots, p_a, p_1),$$

and write $\text{rot}^i p$ for the i -fold rotation.

Lemma 14 (Cyclic-rotation identity). *For every positive p -vector,*

$$3S(p) + D = 2^{p_1} S(\text{rot } p). \quad (3.2)$$

Proof. Using $s_1 = p_1$,

$$\begin{aligned} 3S(p) + D &= 3 \sum_{k=0}^{a-1} 3^{a-1-k} 2^{s_k} + 2^b - 3^a \\ &= \sum_{k=1}^{a-1} 3^{a-k} 2^{s_k} + 2^b \\ &= 2^{p_1} \left(\sum_{k=1}^{a-1} 3^{a-k} 2^{s_k - p_1} + 2^{b-p_1} \right). \end{aligned}$$

The expression in parentheses is $S(\text{rot } p)$. □

Theorem 15 (Exact odd-to-odd bridge). *For $p \in \mathbb{N}_{>0}^a$, the following are equivalent:*

- (i) *there exists $n_0 \in \mathbb{N}_{>0}$ for which (n_0, p) is Collatz-admissible and closed;*
- (ii) *$D = 2^b - 3^a > 0$ and $D \mid S(p)$.*

When these conditions hold, the odd members of the orbit are

$$n_i = \frac{S(\text{rot}^i p)}{D}, \quad i = 0, \dots, a-1. \quad (3.3)$$

Proof. A closed admissible pair gives $Dn_0 = S(p)$ through Proposition 11, and positivity gives $D > 0$.

Conversely, suppose $D > 0$ and $D \mid S(p)$. The integer D is odd. Equation (3.2) therefore implies $D \mid S(\text{rot } p)$, and iteration gives divisibility for every rotation. Each rotated sum is odd: its first term is 3^{a-1} , and all later terms are even. Thus every number in (3.3) is a positive odd integer. Applying (3.2) to $\text{rot}^i p$ and dividing by D gives

$$3n_i + 1 = 2^{p_{i+1}} n_{i+1}.$$

Since n_{i+1} is odd, the valuation is exactly p_{i+1} . Finally $\text{rot}^a p = p$, so the orbit closes. □

Corollary 16 (Rotation invariance of the obstruction). *For every i ,*

$$\gcd(D, S(p)) = \gcd(D, S(\text{rot}^i p)).$$

Hence divisibility, and its failure, are independent of the chosen starting point of a candidate cycle.

Proof. Equation (3.2) and $\gcd(D, 6) = 1$ preserve the gcd after multiplication by 2^{p_1} and 3. Iterate over the rotations. □

Corollary 17 (Cyclic basic class). *The conclusion of Theorem 7 also holds for every cyclic rotation of p_{base} .*

4 The telescoping identity and valid exclusions

Set

$$E_k := 3^{a-1-k} 2^{s_k}, \quad k = 0, \dots, a-1.$$

Thus $S(p) = \sum_k E_k$.

Lemma 18 (Telescoping identity). *For every positive p -vector,*

$$\sum_{k=0}^{a-1} (2^{p_{k+1}} - 3) E_k = 2^b - 3^a = D. \quad (4.1)$$

Consequently,

$$D - S(p) = \sum_{k=0}^{a-1} (2^{p_{k+1}} - 4) E_k. \quad (4.2)$$

Proof. Since $s_{k+1} = s_k + p_{k+1}$,

$$(2^{p_{k+1}} - 3) E_k = 3^{a-1-k} 2^{s_{k+1}} - 3^{a-k} 2^{s_k}.$$

Summing leaves $2^{s_a} - 3^a 2^{s_0} = 2^b - 3^a$. Subtracting $S(p) = \sum E_k$ yields (4.2). $\square$

Proposition 19 (No nontrivial cycle without an entry 1). *Suppose $p_i \geq 2$ for all i . If at least one $p_j \geq 3$, then*

$$0 < S(p) < D,$$

so $D \nmid S(p)$. If every $p_i = 2$, then $S(p) = D$, and the associated closed orbit is the trivial orbit at 1.

Proof. Every term on the right of (4.2) is nonnegative, and a term belonging to $p_j \geq 3$ is positive. This proves $D > S(p) > 0$. For the all-2 vector, every term vanishes, so $D = S(p)$; Theorem 15 then gives $n_i = 1$. $\square$

Proposition 20 (The all-1 vector is expanding). *If every $p_i = 1$, then $b = a$ and $D = 2^a - 3^a < 0$. Hence no positive closed orbit exists.*

Proposition 21 (The cases $a \leq 2$). *If $a \leq 2$, the only positive closed orbit is the trivial orbit.*

Proof. For $a = 1$, $S(p) = 1$, so positivity and $D \mid S(p)$ force $2^{p_1} - 3 = 1$, hence $p_1 = 2$.

For $a = 2$,

$$S(p) = 3 + 2^{p_1}, \quad D = 2^{p_1+p_2} - 9.$$

Positivity gives $b = p_1 + p_2 \geq 4$. If $b = 4$, reduction modulo 7 gives the unique solution $p_1 = p_2 = 2$. If $b \geq 5$, then

$$S(p) \leq 3 + 2^{b-1} < 2^b - 9 = D,$$

so divisibility is impossible. $\square$

5 The mixture with entries 1 and 2

Assume throughout this section that $p_i \in \{1, 2\}$. Define

$$T := \sum_{p_{k+1}=1} E_k, \quad T_2 := \sum_{p_{k+1}=2} E_k.$$

Let t and m be the numbers of entries equal to 1 and 2, respectively. Then

$$a = m + t, \quad b = 2m + t, \quad D = 2^{2m+t} - 3^{m+t}.$$

Lemma 22 (Exact reduction to $D \mid T$). *We have*

$$S(p) = T + T_2, \quad D = T_2 - T, \quad S(p) = D + 2T. \quad (5.1)$$

Since D is odd,

$$D \mid S(p) \iff D \mid T.$$

Proof. In (4.1), the coefficient is -1 for an entry 1 and 1 for an entry 2. Thus $D = T_2 - T$, and the remaining identities follow. Finally $D \mid 2T$ is equivalent to $D \mid T$ because D is odd. $\square$

Lemma 23 (A useful but limited size certificate). *For fixed m, t , the sum T is maximal when all entries 2 precede all entries 1. In that ordering,*

$$T_{\max} = 3^t 2^{2m} \left(1 - \left(\frac{2}{3} \right)^t \right). \quad (5.2)$$

Consequently, if

$$\left(\frac{3}{2} \right)^t \left(1 + \left(\frac{3}{4} \right)^m \right) < 2, \quad (5.3)$$

then $T < D$ for every ordering and $D \nmid S(p)$.

Proof. An adjacent exchange $1, 2 \mapsto 2, 1$ changes the selected contribution by

$$3^{a-2-u} 2^{s_u} (4 - 3) > 0.$$

Repeated exchanges place all 2's first. Summing the resulting geometric series gives (5.2). Comparing it with $D = 2^{2m+t} - 3^{m+t}$ gives (5.3). $\square$

Remark 24. For integral $t \geq 2$, the left side of (5.3) is already greater than 2. Thus this certificate is not a general asymptotic closure mechanism; apart from the all-2 case, it only covers part of the $t = 1$ class.

Proposition 25 (Exactly one entry equal to 1). *Assume $D > 0$. If $t = 1$, then $D \nmid S(p)$.*

Proof. Positivity forces $m \geq 2$, and then $D \geq 5$. The sum T consists of one weight $E_k = 3^{a-1-k} 2^{s_k}$. Since $\gcd(D, 6) = 1$, this weight is a unit modulo D . It cannot be divisible by $D \geq 5$, contradicting Lemma 22. $\square$

Theorem 26 (Exactly two entries equal to 1). *Assume $D > 0$. If $t = 2$, then $D \nmid S(p)$.*

Proof. Let the selected weights be E_i, E_j , with $i < j$, and put $r = j - i$. There are no other selected weights between them, so

$$s_j - s_i = 1 + 2(r - 1) = 2r - 1.$$

Factoring a unit modulo D from $T = E_i + E_j$, the divisibility $D \mid T$ would imply

$$D \mid 3^r + 2^{2r-1}. \quad (5.4)$$

Here

$$a = m + 2, \quad b = 2m + 2, \quad D = 2^{2m+2} - 3^{m+2}.$$

Positivity forces $m \geq 3$, and $1 \leq r \leq m + 1$. For $m \geq 7$,

$$\begin{aligned} 3^r + 2^{2r-1} &\leq 3^{m+1} + 2^{2m+1}, \\ D - (3^{m+1} + 2^{2m+1}) &= 2(4^m - 6 \cdot 3^m) > 0. \end{aligned}$$

Thus the positive number in (5.4) is smaller than D . The remaining finite cases are

m	D	$\{3^r + 2^{2r-1} \bmod D : 1 \leq r \leq m+1\}$
3	13	$\{5, 4, 7, 1\}$
4	295	$\{5, 17, 59, 209, 165\}$
5	1909	$\{5, 17, 59, 209, 755, 868\}$
6	9823	$\{5, 17, 59, 209, 755, 2777, 556\}$

and none of the residues is zero. □

Theorem 27 (Exactly three entries equal to 1). *Assume $D > 0$. If $t = 3$, then $D \nmid S(p)$.*

Proof. Let the selected weights be E_i, E_j, E_k , with $i < j < k$, and put

$$u = j - i, \quad v = k - i.$$

Factoring a unit from T , the divisibility $D \mid T$ would imply

$$D \mid P(u, v) := 3^v + 3^{v-u}2^{2u-1} + 2^{2v-2}, \quad 0 < u < v. \quad (5.5)$$

Now

$$a = m + 3, \quad b = 2m + 3, \quad D = 2^{2m+3} - 3^{m+3}.$$

The condition $D > 0$ is equivalent to $m \geq 5$, and $v \leq m + 2$. For fixed v , the middle term of $P(u, v)$ increases with u , so $P(u, v) \leq P(v - 1, v)$. Moreover,

$$P(v - 1, v) = 3^v + 5 \cdot 2^{2v-3}$$

is increasing in v . Therefore

$$P(u, v) \leq P(m + 1, m + 2) = 3^{m+2} + 5 \cdot 2^{2m+1} =: P_{\max}(m).$$

Both P and D are odd, while

$$P \equiv 1 \pmod{3}, \quad D \equiv 2 \pmod{3}.$$

If $P = qD$, then $q \equiv 5 \pmod{6}$, hence $q \geq 5$. For $m \geq 6$,

$$5D - P_{\max}(m) = 6(5 \cdot 4^m - 24 \cdot 3^m) > 0.$$

Thus $0 < P < 5D$, a contradiction. For $m = 5$, $D = 1631$ and $P \leq 12427 < 11D$, so only $q = 5$ remains. If $v \leq 6$, then $P \leq P(5, 6) = 3289 < 5D$. If $v = 7$, strict monotonicity in u gives

$$P(1, 7) = 7741 < 5D = 8155 < 8227 = P(2, 7),$$

so equality is impossible there as well. □

Corollary 28 (Scope of the small- t results). *Within $p_i \in \{1, 2\}$, a nontrivial positive candidate satisfying $D \mid S(p)$ must have $t \geq 4$.*

Remark 29. This corollary identifies the first class not covered by the present small- t lemmas. It is not a proposal to settle $t = 4, 5, 6, \dots$ one at a time. A complete argument would need a statement uniform in t , or a finite structural division covering all t .

6 Cyclic gaps and the limit of uniform size estimates

Continue to assume $p_i \in \{1, 2\}$, $D > 0$, and $t \geq 1$. List the cyclic distances between consecutive entries equal to 1 as

$$g_1, \dots, g_t \geq 1, \quad g_1 + \dots + g_t = a.$$

For a cut that omits g_t , put

$$d_0 = 0, \quad d_h = g_1 + \dots + g_h \quad (1 \leq h \leq t-1), \quad L = a - g_t,$$

and define

$$P(g_1, \dots, g_{t-1}) := \sum_{h=0}^{t-1} 3^{L-d_h} 2^{2d_h-h}. \quad (6.1)$$

Write P_c for the value obtained from the c -th cyclic cut.

Proposition 30 (Normalized gap reduction). *For every cyclic cut,*

$$D \mid T \iff D \mid P_c.$$

Proof. First rotate the word so that the chosen cut becomes the wrap-around cut. By Corollary 16 and Lemma 22, this rotation preserves the equivalence $D \mid T$. Let

$$i_0 < i_1 < \dots < i_{t-1}$$

be the selected weight indices in the rotated linear word. Then $d_h = i_h - i_0$. Among the d_h entries contributing to $s_{i_h} - s_{i_0}$, exactly h equal 1 and $d_h - h$ equal 2. Hence

$$s_{i_h} - s_{i_0} = h + 2(d_h - h) = 2d_h - h.$$

Factoring $3^{a-1-i_{t-1}} 2^{s_{i_0}}$ from the rotated selected sum T leaves precisely (6.1). The factored number is a unit modulo D , because $\gcd(D, 6) = 1$. $\square$

Lemma 31 (Multiplier congruence). *If $D \mid P_c$ and $P_c = q_c D$, then*

$$q_c \equiv 5 \pmod{6}.$$

In particular, $q_c \geq 5$.

Proof. Both P_c and D are odd. Modulo 3, only the final term of P_c survives, and

$$P_c \equiv (-1)^{t-1} \pmod{3}, \quad D \equiv (-1)^t \pmod{3}.$$

Thus $q_c \equiv 2 \pmod{3}$. Combining this with oddness gives $q_c \equiv 5 \pmod{6}$. $\square$

Proposition 32 (Exact failure of the bound $P_c < 5D$). *Let*

$$p = (1, 2)^4 (1, 2, 2)^3.$$

Then

$$(a, m, t, b) = (17, 10, 7, 27), \quad D = 2^{27} - 3^{17} = 5\,077\,565,$$

and the cyclic gaps are

$$(2, 2, 2, 2, 3, 3, 3).$$

The seven normalized sums are

$$\begin{array}{cccc} 26\,892\,097, & 91\,395\,523, & 103\,454\,659, & \\ 117\,021\,187, & 132\,283\,531, & 37\,363\,417, & 31\,684\,057. \end{array}$$

Consequently,

$$\min_c P_c = 26\,892\,097 > 5D = 25\,387\,825.$$

Nevertheless,

$$S(p) = 489\,135\,311, \quad S(p) \bmod D = 1\,689\,071,$$

so this vector is not a cycle.

Theorem 33 (No universal constant after cyclic optimization). *For every fixed real constant $C > 0$, there exist parameters m, t and cyclic gaps $g_i \in \{2, 3\}$ such that $D > 0$ and*

$$\min_c P_c > CD.$$

Proof. Let $\alpha = \log_2 3$. Since α is irrational, it has infinitely many upper continued-fraction convergents

$$\frac{b_\nu}{a_\nu} > \alpha, \quad \delta_\nu = b_\nu - a_\nu \alpha > 0, \quad \delta_\nu \rightarrow 0.$$

For all sufficiently large ν , $3/2 < b_\nu/a_\nu < 5/3$. Put

$$t_\nu = 2a_\nu - b_\nu, \quad m_\nu = b_\nu - a_\nu.$$

Then $a_\nu = m_\nu + t_\nu$ and $b_\nu = 2m_\nu + t_\nu$. Choose

$$N_3 = 2b_\nu - 3a_\nu, \quad N_2 = 5a_\nu - 3b_\nu$$

gaps of lengths 3 and 2, respectively. These counts are nonnegative and satisfy

$$N_2 + N_3 = t_\nu, \quad 2N_2 + 3N_3 = a_\nu.$$

Every omitted gap is at most 3, so the first term of every cut sum is at least $3^{a_\nu-3}$. Hence

$$\min_c P_c \geq 3^{a_\nu-3}.$$

On the other hand,

$$D_\nu = 2^{b_\nu} - 3^{a_\nu} = 3^{a_\nu}(2^{\delta_\nu} - 1),$$

and therefore

$$\frac{\min_c P_c}{D_\nu} \geq \frac{1}{27(2^{\delta_\nu} - 1)} \rightarrow \infty.$$

The required strict inequality follows for sufficiently large ν . □

Remark 34 (Logical scope). Theorem 33 concerns candidate gap configurations. It does not produce a vector satisfying $D \mid S(p)$, and it does not disprove a stronger conditional argument that uses divisibility or additional local arithmetic. It proves that cyclic placement and a fixed Archimedean constant alone cannot close the general class.

Proposition 35 (Largest-gap bracket). *Let $G = \max_i g_i$. Then*

$$3^{a-G} \leq \min_c P_c < 2 \cdot 4^{a-G}. \tag{6.2}$$

Consequently,

$$2 \cdot 4^{a-G} \leq 5D \implies D \nmid T. \tag{6.3}$$

Proof. For the cut P_c , its first term is $3^{a-g_{c-1}} \geq 3^{a-G}$, because this cut omits g_{c-1} . For a cut omitting a gap of length G , put $L = a - G$. Then

$$\begin{aligned} P_c &= 3^L \sum_{h=0}^{t-1} \frac{(4/3)^{d_h}}{2^h} \\ &\leq 3^L (4/3)^L \sum_{h=0}^{t-1} 2^{-h} < 2 \cdot 4^L. \end{aligned}$$

This proves (6.2). If $D \mid T$, then Proposition 30 and Lemma 31 would give $P_c \geq 5D$ for every cut, contradicting (6.3). $\square$

Proposition 36 (Neighboring-cut identity). *Index the gaps cyclically. Let P_c begin at the c -th selected entry, include*

$$g_c, g_{c+1}, \dots, g_{c+t-2},$$

and omit g_{c-1} . Then

$$2^{2g_{c-1}} P_{c+1} - 3^{g_{c-1}} P_c = D. \quad (6.4)$$

If $D \mid P_c$, then all cuts are divisible by D , and writing $P_c = q_c D$ gives the cyclic system

$$2^{2g_{c-1}} q_{c+1} - 3^{g_{c-1}} q_c = 1, \quad q_c \equiv 5 \pmod{6}. \quad (6.5)$$

Proof. Expand the two sums from (6.1). After multiplying P_{c+1} by $2^{2g_{c-1}}$, all interior terms match the terms of $3^{g_{c-1}} P_c$. The two endpoint terms leave $2^{2m+t} - 3^{m+t} = D$. Divisibility propagates through (6.4) because D is odd; the quotient system then follows from Lemma 31. $\square$

7 Neighboring quotients and finite-window certificates

This section records consequences of (6.5). Every statement is conditional on a positive integer quotient cycle unless explicitly described as a formal local-window calculation. In particular, none of the finite computations below is used as an asymptotic theorem.

7.1 Exact consequences of the quotient system

Put

$$A_c = 2^{2g_{c-1}}, \quad B_c = 3^{g_{c-1}}.$$

Then (6.5) is

$$A_c q_{c+1} - B_c q_c = 1. \quad (7.1)$$

Proposition 37 (Exact scope of the quotient system). *Unrolling all t equations from any cut c gives*

$$(2^b - 3^a) q_c = P_c. \quad (7.2)$$

Consequently, the cyclic rational solution is unique and equals $q_c = P_c/D$. A positive integer quotient cycle exists if and only if the original divisibility condition $D \mid P_c$ holds.

Proof. Successively substitute (7.1) into itself around the cycle. The coefficient of the terminal copy of q_c is $\prod_c A_c = 2^b$, the coefficient of the initial copy is $\prod_c B_c = 3^a$, and the accumulated inhomogeneous term is exactly the normalized cut sum P_c . This gives (7.2). Since $D > 0$, uniqueness and the integrality equivalence follow. $\square$

Proposition 38 (Monotonicity, coprimality, and global identities). *For every index c ,*

$$q_{c+1} > q_c \iff B_c > A_c, \quad (7.3)$$

$$\gcd(q_c, q_{c+1}) = 1. \quad (7.4)$$

Moreover,

$$\sum_{c=0}^{t-1} (2^{2g_c-1} - 3^{g_c}) q_{c+1} = t, \quad (7.5)$$

and

$$\boxed{\frac{2^b}{3^a} = \prod_{c=0}^{t-1} \left(1 + \frac{1}{3^{g_{c-1}} q_c}\right)}. \quad (7.6)$$

Proof. Subtract q_c from (7.1) divided by A_c :

$$q_{c+1} - q_c = \frac{1 - (A_c - B_c)q_c}{A_c}.$$

The two coefficients are unequal powers of distinct primes and $q_c \geq 5$, which proves (7.3). Any common divisor of adjacent quotients divides the left-hand side of (7.1), proving (7.4). Summing (7.1) and shifting the second index gives (7.5). Finally, divide (7.1) by $B_c q_c$, multiply over the cycle, and cancel the cyclic product of quotients. $\square$

For gaps in $\{2, 3\}$, a current gap 2 is therefore a strict rise and a current gap 3 a strict fall. Quotient minima lie at transitions $3 \rightarrow 2$, and maxima at transitions $2 \rightarrow 3$.

Corollary 39 (Conditional gap-density corridor). *Every positive integer quotient cycle with $t \geq 2$ satisfies*

$$\frac{\log 2}{\log(4/3)} < \frac{a}{t} \leq \frac{\log(2\sqrt{544/495})}{\log(4/3)}. \quad (7.7)$$

Numerically,

$$2.4094208397 < \frac{a}{t} \leq 2.5734760474.$$

Proof. The strict lower bound is equivalent to $2^b > 3^a$, using $b = 2a - t$. For the upper bound, note that $B_c \geq 3$ and $q_c \geq 5$. By (7.4), two adjacent quotients cannot both equal 5; every other admissible quotient is at least 11. Hence (7.6) gives

$$\frac{2^b}{3^a} \leq \left(\frac{16}{15}\right)^{\lfloor t/2 \rfloor} \left(\frac{34}{33}\right)^{\lceil t/2 \rceil}.$$

The displayed uniform bound follows after taking logarithms. For odd t , the same calculation is slightly stronger. $\square$

7.2 The CRT window hierarchy and its limitation

Let $K_c = A_c - B_c$. Since all quotients are 5 (mod 6), equation (7.1) implies

$$q_c \equiv K_c^{-1} \pmod{6A_c}, \quad q_{c+1} \equiv K_c^{-1} \pmod{6B_c}. \quad (7.8)$$

The inverses exist because K_c is a unit modulo both displayed moduli. Combining the outgoing congruence at c with the incoming congruence at $c - 1$ gives one class

$$q_c \equiv \rho_c \pmod{2^{2g_c} 3^{g_{c-2}+1}}. \quad (7.9)$$

For gaps in $\{2, 3\}$, the eight possibilities are:

(g_{c-2}, g_{c-1}, g_c)	forced class for q_c
222	431 (mod 432)
223	1511 (mod 1728)
232	101 (mod 432)
233	1613 (mod 1728)
322	1151 (mod 1296)
323	4391 (mod 5184)
332	389 (mod 1296)
333	1037 (mod 5184)

In particular,

$$q_c \geq 101 \quad \text{for every } c. \quad (7.10)$$

The construction extends to arbitrary window depth. Define

$$\begin{aligned} A_{c,k} &= \prod_{j=0}^{k-1} A_{c+j}, & B_{c,k} &= \prod_{j=0}^{k-1} B_{c+j}, \\ C_{c,0} &= 0, & C_{c,k+1} &= B_{c+k}C_{c,k} + A_{c,k}. \end{aligned}$$

Then

$$A_{c,k}q_{c+k} - B_{c,k}q_c = C_{c,k}. \quad (7.11)$$

The forward and backward windows determine compatible congruences

$$q_c \equiv C_{c,k}(A_{c,k} - B_{c,k})^{-1} \pmod{6A_{c,k}}, \quad (7.12)$$

$$q_c \equiv C_{c-k,k}(A_{c-k,k} - B_{c-k,k})^{-1} \pmod{6B_{c-k,k}}. \quad (7.13)$$

Let $\rho_c^{(k)}$ be the least positive representative of their CRT combination. For a fixed radius- k gap block this is exactly the least positive center realizing the whole integer path, not merely a formal endpoint residue. Indeed, the composed 2-adic divisibility peels off one transition at a time; endpoint oddness makes every intermediate quotient odd and hence 5 (mod 6). The backward window works identically, and positivity follows from $q \geq 5$.

Equation (7.6) yields the necessary condition

$$\frac{2^b}{3^a} \leq U_k(g) := \prod_c \left(1 + \frac{1}{3^{g_{c-1}} \rho_c^{(k)}} \right). \quad (7.14)$$

A strict reverse inequality is an exact certificate that the candidate word cannot close.

Proposition 40 (No fixed window depth is uniform). *For every fixed k , there are arbitrarily long formal gap words over $\{2, 3\}$, with $D > 0$, that are not excluded by (7.14).*

Proof. Use block words $2^{N_2}3^{N_3}$ associated with upper continued- fraction convergents $b/a \rightarrow \log_2 3$. Then $2^b/3^a \rightarrow 1^+$. For fixed k , every position sufficiently deep inside either homogeneous run has one fixed finite representative $\rho^{(k)}$. Thus $U_k(g)$ contains a linearly growing number of fixed factors strictly greater than 1, while its left-hand side approaches 1. Hence the required reverse inequality eventually fails. $\square$

At the other extreme, when $k = t$, the composed constant is exactly $C_{c,t} = P_c$, so the full-window condition is equivalent to $D \mid P_c$. The hierarchy is therefore complete at full length but then merely rewrites the original divisibility problem.

7.3 Primitive words and the partial state dynamics

Proposition 41 (Reduction to a primitive gap word). *Let a cyclic gap word be a proper power w^r , and put $A = 2^{b_0}$, $B = 3^{a_0}$ for the base word w . With*

$$G_r = \sum_{j=0}^{r-1} A^j B^{r-1-j},$$

one has, for corresponding cuts,

$$D(w^r) = D(w)G_r, \quad P_c(w^r) = P_c(w)G_r.$$

Consequently,

$$D(w^r) \mid P_c(w^r) \iff D(w) \mid P_c(w).$$

Proof. The denominator identity is the standard factorization of $A^r - B^r$. Repeating the affine block gives the same geometric factor in its inhomogeneous term, which is the corresponding normalized cut sum. $\square$

For the alphabet $\{2, 3\}$, write a state as

$$(h, q) = (g_{c-1}, q_c), \quad h \in \{2, 3\}, \quad q \equiv 5 \pmod{6}.$$

The next gap is forced by

$$2g_c - 1 = v_2(3^h q + 1).$$

The only branches, for $n \geq 0$, are

$$\begin{aligned} (2, 47 + 48n) &\mapsto (2, 53 + 54n), \\ (2, 167 + 192n) &\mapsto (3, 47 + 54n), \\ (3, 5 + 48n) &\mapsto (2, 17 + 162n), \\ (3, 77 + 192n) &\mapsto (3, 65 + 162n). \end{aligned} \tag{7.15}$$

Every state has at most one predecessor as well as at most one successor. In the reverse direction it is essential not only that the relevant 3-adic valuation equal 2 or 3, but also that the quotient left after division be $2 \pmod{3}$.

Let $\mathcal{S}_k$ be the states admitting k predecessors and k successors, and define

$$m_k = \min\{q : (h, q) \in \mathcal{S}_k\}.$$

The radius- k state classes have common period

$$L_k = 2^{1+5k} 3^{1+3k} = 6 \cdot 864^k. \tag{7.16}$$

Proposition 42 (Exact state density and the bi-infinite obstruction). *Modulo L_k , exactly $2 \cdot 20^k$ admissible state classes survive k steps in both directions. Their relative density among states with $h \in \{2, 3\}$ and $q \equiv 5 \pmod{6}$ is*

$$\left(\frac{5}{216}\right)^k. \tag{7.17}$$

Moreover, m_k is nondecreasing and

$$(m_k)_{k \geq 1} \text{ is bounded} \iff \text{a positive bi-infinite integer quotient orbit exists.} \tag{7.18}$$

Proof. Fix the central value of h . Lifting all future gap words to the largest 2-power in L_k gives total multiplicity $(4 + 1)^k = 5^k$: a future gap 2 loses two powers of 2 relative to a gap 3. Similarly, the past words contribute $(3 + 1)^k = 4^k$ lifts in the 3-part. The two central values of h give $2 \cdot 5^k 4^k = 2 \cdot 20^k$ classes. They are disjoint by the deterministic state map. Division by the $2(L_k/6)$ admissible ambient state classes proves (7.17).

The sets $\mathcal{S}_{k+1} \subseteq \mathcal{S}_k$, so m_k is nondecreasing. If the minima are bounded, one of finitely many states under that bound survives arbitrarily large radii. Uniqueness in both directions yields one bi-infinite orbit. The converse is immediate. $\square$

Thus proving $m_k \rightarrow \infty$ would be stronger than excluding only periodic quotient cycles: it would also exclude nonperiodic bi-infinite paths of (7.15). The exact density (7.17) does not itself control the least exceptional residue.

Proposition 43 (A repeated fixed noncycle grows exponentially). *Suppose one fixed motif composes to*

$$Aq' - Bq = N, \quad A = 2^\beta, \text{quad} B = 3^\alpha, \text{quad} A - B > 0, \text{quad} N > 0.$$

For windows aligned with r exact motif repetitions on each side, the center satisfies

$$(A - B)q \equiv N \pmod{6(AB)^r}. \quad (7.19)$$

If $(A - B) \nmid N$ and $6(AB)^r > N$, then

$$q \geq \frac{6(AB)^r - N}{A - B}. \quad (7.20)$$

If $(A - B) \mid N$, the quotient $N/(A - B)$ is precisely the base integer-cycle case.

Proof. Iterate the affine motif in both directions and impose the endpoint conditions $q \equiv 5 \pmod{6}$. Since $\gcd(A - B, 6AB) = 1$, the resulting two congruences combine to (7.19). If $(A - B) \nmid N$, write $(A - B)q = N + \ell 6(AB)^r$. Once the modulus exceeds N , positivity forces $\ell \geq 1$, proving (7.20). The divisible case is the fixed quotient of the base motif. $\square$

Remark 44 (Exact finite computation). The first selected values of m_k , computed with exact integer CRT arithmetic, are

k	m_k
1	101
2	9,173
4	19,118,591
8	118,204,070,363,135
12	48,146,916,957,781,546,325
16	537,279,696,046,626,464,722,054,349
20	104,020,091,111,386,500,663,723,752,248,661

The meet-in-the-middle calculation was cross-checked against exhaustive enumeration through $k = 10$. These are finite data, not an asymptotic lower bound. The companion scripts `research_neighbor_quotients.py` and `research_window_mitm.py` reproduce the identities and the reported window minima.

8 The logarithmic bridge for a hypothetical cycle

The statements in this section apply only after an exact positive cycle has been assumed. They do not hold for an arbitrary formal candidate.

Proposition 45 (Cycle-product identity). *Let $n_0, \dots, n_{a-1}$ be a positive odd cycle with exponent vector p , read cyclically. Then*

$$2^b = 3^a \prod_{i=0}^{a-1} \left(1 + \frac{1}{3n_i}\right). \quad (8.1)$$

Equivalently, with

$$\Lambda = b \log 2 - a \log 3,$$

we have

$$\Lambda = \sum_{i=0}^{a-1} \log \left(1 + \frac{1}{3n_i}\right). \quad (8.2)$$

Proof. Multiply the exact cycle equations

$$2^{p_{i+1}} n_{i+1} = 3n_i + 1 = 3n_i \left(1 + \frac{1}{3n_i}\right)$$

over all i . The cyclic products of the n_i 's cancel. Taking logarithms gives (8.2). $\square$

Corollary 46 (Archimedean height bounds). *Let*

$$n_{\min} = \min_i n_i, \quad n_{\max} = \max_i n_i.$$

Then

$$\frac{a}{3n_{\max} + 1} \leq \Lambda < \frac{a}{3n_{\min}}, \quad (8.3)$$

and hence

$$0 < \frac{b}{a} - \log_2 3 < \frac{1}{3n_{\min} \log 2}. \quad (8.4)$$

Proof. For $x > 0$,

$$\frac{x}{1+x} \leq \log(1+x) < x.$$

Apply these inequalities termwise in (8.2). $\square$

Corollary 47 (When minimal contraction is forced). *Put*

$$\theta = \{a \log_2 3\}, \quad \mathcal{U} = \frac{a}{3n_{\min} \log 2}.$$

For a positive cycle there is an integer $\ell \geq 1$ such that

$$b - a \log_2 3 = \ell - \theta, \quad 0 < \ell - \theta < \mathcal{U}.$$

Therefore:

- (i) *if $\mathcal{U} \leq 1 - \theta$, no such cycle exists;*
- (ii) *if $\mathcal{U} \leq 2 - \theta$ and a cycle exists, then*

$$b = \lfloor a \log_2 3 \rfloor + 1.$$

Corollary 48 (Continued-fraction reduction). *If*

$$n_{\min} > \frac{2a^2}{3 \log 2},$$

then the reduced fraction represented by b/a is a continued-fraction convergent of $\log_2 3$.

Proof. Equation (8.4) gives

$$\left| \frac{b}{a} - \log_2 3 \right| < \frac{1}{3n_{\min} \log 2} < \frac{1}{2a^2}.$$

After reducing b/a , its denominator is at most a , so Legendre's criterion applies. $\square$

8.1 Run bounds at the global quotient minimum

The next statement returns to a hypothetical integer solution of the neighboring-quotient system with gaps in $\{2, 3\}$.

Theorem 49 (Adjacent runs at a global quotient minimum). *Let*

$$Q = \min_c q_c.$$

Let u be the length of the maximal run of gaps 2 immediately following Q , and v the length of the maximal run of gaps 3 immediately preceding Q . Then

$$Q \geq \frac{2^{3u+1} - 1}{3}, \quad Q \geq \frac{3^{3v-1} + 1}{5}. \quad (8.5)$$

There are effectively computable absolute constants $C_0, C_1 > 0$ such that

$$u \leq C_0 + C_1 \log t, \quad v \leq C_0 + C_1 \log t. \quad (8.6)$$

Proof. By (7.3), the central transition is necessarily $3 \rightarrow 2$. Iteration through the following 2-run gives

$$q_u + 1 = \frac{9^u(3Q + 1)}{8^u}. \quad (8.7)$$

Since $q_u \equiv 5 \pmod{6}$, the integer $(3Q + 1)/8^u$ is positive and even. Thus $3Q + 1 \geq 2 \cdot 8^u$, proving the first inequality in (8.5). Backward iteration through the preceding 3-run gives

$$32^v(5Q - 1) = 3^{3v-1}(5q_{-v} - 3). \quad (8.8)$$

Hence $3^{3v-1} \mid 5Q - 1$, which proves the second inequality.

Put $\Lambda = b \log 2 - a \log 3$. Since every gap is at least 2 and every quotient is at least Q ,

$$\Lambda = \sum_c \log \left(1 + \frac{1}{3^{g_{c-1} q_c}} \right) < \frac{t}{9Q}. \quad (8.9)$$

Matveev's explicit lower bound for a nonzero linear form in logarithms [4], applied to $b \log 2 - a \log 3$ with $a \leq 3t$ and $b \leq 5t$, gives an effectively computable constant $K > 0$ for which

$$\log \Lambda > -K(1 + \log(5t)).$$

Thus $\Lambda \geq ct^{-\kappa}$ for effective absolute $c, \kappa > 0$, and (8.9) gives a polynomial upper bound for Q . Comparing that bound with the two exponential lower bounds in (8.5) proves (8.6). $\square$

Corollary 50 (Effective finiteness of the two-run class). *There is an effectively computable bound T_0 such that no cyclic two-run word $2^u 3^v$ with $u + v > T_0$ can have a positive integer quotient cycle.*

Proof. For a two-run cyclic word, the runs in Theorem 49 contain the entire word, so $t = u + v \leq 2C_0 + 2C_1 \log t$. This inequality has only finitely many positive integer solutions and yields an effective bound. $\square$

Remark 51. The explicit bound supplied by a general linear-form estimate is very large and has not been instantiated or exhaustively checked here. The corollary is an effective finite reduction, not a completed exclusion of every two-run word. The comparison between exponential run bounds and linear forms is methodologically related to standard m -cycle estimates such as [3]; those results do not by themselves prove the quotient statement above. For a general word, Theorem 49 controls only the two runs adjacent to the global quotient minimum.

9 Residual arithmetic problems

For a general positive p -vector, define

$$A(p) := \sum_{p_{k+1}=1} E_k, \quad R(p) := \sum_{p_{k+1} \geq 3} (2^{p_{k+1}} - 4) E_k.$$

The exact signed remainder is

$$D - S(p) = R(p) - 2A(p). \quad (9.1)$$

Proposition 52 (Rotated size certificate). *Assume $D > 0$. If some cyclic rotation $p' = \text{rot}^i p$ satisfies*

$$R(p') > 2A(p'),$$

then $D \nmid S(p)$.

Proof. Equation (9.1) gives $0 < S(p') < D$, so $D \nmid S(p')$. Corollary 16 transfers the obstruction to p . $\square$

Remark 53 (The certificate is not universal). For

$$p = (3, 1)^r,$$

one has

$$D_r = 16^r - 9^r, \quad S_r = 11 \sum_{j=0}^{r-1} 16^j 9^{r-1-j} = \frac{11}{7} D_r > D_r.$$

Thus entries equal to 1 can dominate strong declines at arbitrary length. This family refutes a universal size argument; it is not a family of cycles, because $11/7 \notin \mathbb{N}_{>0}$.

Let $q^e \parallel D$. Since $\gcd(D, 6) = 1$, define the exact local residue

$$\Phi_{p, q^e} := \sum_{k=0}^{a-1} 2^{s_k} 3^{-k} \pmod{q^e}. \quad (9.2)$$

Then

$$S(p) \equiv 3^{a-1} \Phi_{p, q^e} \pmod{q^e}.$$

In the $\{1, 2\}$ class, the selected residue

$$\Psi_{p, q^e} := \sum_{p_{k+1}=1} 2^{s_k} 3^{-k} \pmod{q^e} \quad (9.3)$$

satisfies

$$T \equiv 3^{a-1} \Psi_{p, q^e} \pmod{q^e}.$$

Thus one nonzero local residue is a valid short certificate of failed global divisibility. Large multiplicative orders alone, however, do not prove that these additive sums are nonzero.

9.1 Affine prime-power certificates

Let M be an odd modulus with $\gcd(M, 6) = 1$. For a gap g , put

$$u_g = 3^g 2^{-(2g-1)} \pmod{M}, \quad f_g(z) = 1 + u_g z. \quad (9.4)$$

For a linearly written cyclic word $w = (g_0, \dots, g_{t-1})$, apply the maps in the listed order and write

$$F_w = f_{g_{t-1}} \circ \dots \circ f_{g_0}, \quad F_w(z) = C_w + U_w z. \quad (9.5)$$

Its multiplier is

$$U_w = 3^a 2^{-b} \pmod{M}.$$

Proposition 54 (Exact affine modular criterion). *If $M \mid D$, then $U_w = 1$, and*

$$M \mid P_c \text{ for one, equivalently every, cut} \iff F_w = \text{id} \pmod{M}. \quad (9.6)$$

Proof. Normalize a partial cut sum by its last power-of-two term. The state starts at 1, and appending a gap g applies exactly the affine map f_g . Thus its terminal state is P_c multiplied by a unit modulo M . Rotate the word so that the omitted gap is composed last. When $M \mid D$, the full multiplier is 1, and the full affine translation is zero exactly when the cut state is zero. Cyclic rotations of F_w are conjugate because every f_g is invertible; being the identity is therefore rotation-invariant. $\square$

Proposition 55 (Closed criterion modulo 5 for gaps $\{2, 3\}$). *Let n_2 be the number of gaps 2. Then*

$$5 \mid D \iff n_2 \equiv 0 \pmod{4}. \quad (9.7)$$

For a chosen linear cut, define

$$T_w = \sum_{i: g_i=3} 3^{\#\{j: i < j < t, g_j=2\}} \pmod{5}. \quad (9.8)$$

If $n_2 \equiv 0 \pmod{4}$, then

$$5 \mid P_c \text{ for every cut} \iff T_w = 0. \quad (9.9)$$

Proof. Modulo 5, one has $u_2 = 3$ and $u_3 = 1$, which proves (9.7). Conjugate the affine coordinate by $y = z - 2$. Then

$$f_2 : y \mapsto 3y, \quad f_3 : y \mapsto y + 1.$$

Every gap 3 contributes one translation, multiplied by 3 once for each later gap 2. This gives (9.8). When $3^{n_2} = 1$, the coordinate shift does not change the full translation, so Proposition 54 proves (9.9). $\square$

Corollary 56 (An infinite structural mod-5 exclusion). *Suppose every cyclic run of gaps 2 has length divisible by 4. If the number n_3 of gaps 3 is not divisible by 5, then the word cannot have an integer quotient cycle.*

Proof. Choose a linear cut at a run boundary. Every exponent in (9.8) is then divisible by 4, so $T_w \equiv n_3 \not\equiv 0 \pmod{5}$. Rotation-invariance completes the argument. $\square$

For example, the primitive word 22223333 has

$$D = 808\,182\,895, \quad v_5(D) = 1,$$

while the cut beginning at the first 2 has

$$P = 860\,304\,347 \equiv 2 \pmod{5}.$$

Proposition 57 (A prime-power family over $\{2, G\}$). *Fix $G \geq 3$, let*

$$H_G = 2^{2G-1} - 3^G, \quad p^e \parallel H_G, \quad d = \text{ord}_{p^e}(9/8).$$

Every cyclic word over $\{2, G\}$ whose runs of 2's all have length divisible by d satisfies

$$F_w(z) = z + n_G \pmod{p^e}, \quad (9.10)$$

where n_G is the number of gaps G . Consequently, if $p \nmid n_G$, then $p^e \mid D$ but $p^e \nmid P_c$, and the word is excluded.

In particular, consider the two-run word consisting of dr copies of 2 followed by k copies of G . If $p \nmid k$ and $p \mid r$, then $p^e \parallel D$. Choosing k/r sufficiently large gives $D > 0$, and every such mixed two-run cyclic word is primitive.

Proof. For $x = 9/8$, the relation $x^d = 1 \pmod{p^e}$ and the fact that $x - 1 = 1/8$ is a unit show that the full affine iterate f_2^d is the identity, including its geometric translation. Since $p^e \mid H_G$, one has $f_G(z) = z + 1$. Removing all 2-runs therefore leaves exactly n_G translations, proving (9.10) and the obstruction.

For the valuation assertion, write $y = 3^G/2^{2G-1}$. Then $v_p(y - 1) = e$. Odd-prime LTE gives $v_p(y^k - 1) = e$ when $p \nmid k$, whereas $v_p(x^{dr} - 1) \geq e + 1$ when $p \mid r$. Their product differs from 1 to exact order e , so $v_p(D) = e$. A sufficiently large proportion of gaps G makes the real multiplier expansive, hence $D > 0$. Finally, a mixed two-run cyclic word has exactly two symbol transitions; a nonconstant proper power has at least four. $\square$

Proposition 58 (Limitation of fixed modular lists). *No fixed finite list of affine modular-identity tests can reject every primitive formal $\{2, 3\}$ candidate with $D > 0$.*

Proof. Take the least common multiple M of the fixed odd moduli and let L_2, L_3 be the orders of f_2, f_3 in the finite affine group modulo M . Every two-run word with L_2r copies of 2 and L_3k copies of 3 has $F_w = \text{id} \pmod{M}$, so all listed tests accept it. Choose k/r large enough to have $D > 0$. The word is primitive by the two-transition argument, and infinitely many choices remain. $\square$

This proposition concerns only a fixed finite list of these modular identity obstructions. It does not rule out an adaptive modulus depending on the word, nor a proof combining modular and Archimedean information.

Remark 59 (Why full prime powers are necessary). For the all-3 word of length t ,

$$D = 32^t - 27^t, \quad P_c = \frac{D}{5}.$$

If $5 \mid t$, then LTE gives $v_5(D) = 1 + v_5(t)$. Every prime divisor of D divides P_c , although $D \nmid P_c$; the missing factor is detected only by the full 5-power. This word is already excluded by monotonicity, but it shows why a uniform local lemma must use $p^e \parallel D$, not only the radical of D .

Open problem 60 (Uniform local obstruction). For every unresolved positive p -vector, find a non-circular rule selecting a prime power $q^e \parallel D$ and prove

$$\Phi_{p,q^e} \not\equiv 0 \pmod{q^e},$$

or, in the $\{1, 2\}$ reduction,

$$\Psi_{p,q^e} \not\equiv 0 \pmod{q^e}.$$

The proof must control additive cancellation along the monotone path (s_k, k) . Propositions 55 and 57 solve infinite structural subfamilies only; the selector here must cover every remaining primitive candidate.

Open problem 61 (Uniform use of the neighboring quotients). Combine the system (6.5) with prime-power or valuation constraints to exclude all $\{1, 2\}$ configurations with $t \geq 4$. One possible sufficient route is a uniform proof that $m_k \rightarrow \infty$, or a growing-window bound strong enough for (7.14), followed by an adaptive prime-power obstruction for the exceptional words. A solution must be uniform in t , or reduce all t to finitely many structural regimes; checking successive fixed values of t is not a closure argument.

Open problem 62 (Height separation). For an unresolved class satisfying $D \mid S(p)$, derive an effective lower bound on $n_{\min}$, or an equivalently strong upper bound on $\Lambda = b \log 2 - a \log 3$, that can be combined with (8.4) and explicit Diophantine estimates.

10 Proved and open inventory

The following statements are proved here or explicitly retained with their limited scope:

- the formal affine formula, its corrected extrema, and the finiteness of the formal fixed-point set;
- the coherent p -vector correction term;
- the exact affine composition, cyclic-rotation identity, and odd-to-odd divisibility bridge;
- the established basic-fractal result in its original cyclic class;
- exclusion of every vector with no entry 1, apart from the trivial all-2 orbit;
- exclusion of $a \leq 2$ and of the $\{1, 2\}$ classes with exactly one, two, or three entries equal to 1;
- failure of every fixed-constant cyclic size strategy, together with the valid largest-gap and neighboring-cut identities;
- the exact quotient product, density corridor, CRT window hierarchy, partial state map, and survivor-density formula;
- effective finite reduction of the cyclic two-run class $2^u 3^v$, without claiming that its resulting finite range has been exhausted;
- affine prime-power certificates, including the complete mod-5 criterion and infinite primitive exclusion families;
- the exact product and logarithmic identities required of any hypothetical positive cycle.

The following tasks remain open in this framework:

- the $\{1, 2\}$ class with $t \geq 4$ in a uniform form;
- a uniform lower bound for the least growing-window residue, or an adaptive prime-power selector covering the residual primitive gap words;
- mixed vectors containing both entries 1 and entries at least 3, unless a displayed size or local certificate applies;
- exclusion of all nontrivial positive Collatz cycles;
- exclusion of unbounded positive trajectories, which is a logically separate problem.

Finite computation can verify displayed arithmetic and explore residual classes, but it cannot turn a bounded search into any of the open uniform theorems above. Accordingly, no full proof of the Collatz conjecture is claimed.

References

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